Influence of the Accuracy of Geometrical Modeling with Large Curvilinear Elements on FEM Solutions to EM Problems

Anđelija Ž. Ilić, *Member, IEEE*, Milan M. Ilić, *Member, IEEE*, and Branislav M. Notaroš, *Senior Member, IEEE*

***Abstract* — The accuracy of the large, curvilinear, Lagrange-type elements has been analyzed. We have compared the results of the four differently *hp*-refined models and pointed out the accuracy of our models and the necessity for the large-domain approach in modeling. The influence of geometrical inexactness to the limit of achievable accuracy has been investigated. It has been shown that the elements used here represent a good choice for fast and reliable geometrical modeling of EM structures.**

***Keywords* — curvilinear, finite element method (FEM), geometrical modeling, geometrically higher order elements, *hp*-refinement, large-domain approach.**

I. Introduction

THE modeling of electromagnetic structures by the finite elements of higher (arbitrary) orders offers significant advantages compared to the traditionally employed first (and sometimes second) order elements. In our previous work [1], [2], we have shown that these advantages can not be fully exploited within the small-domain modeling techniques (utilization of the electrically very small geometrical elements, typically on the order of λ/10 in each dimension, λ being the wavelength in the medium). In contrast to the small-domain techniques, the large-domain computational approach (electrically large geometrical elements – typically on the order of λ in each dimension – are used to model electromagnetic structures) can greatly reduce the number of unknowns for a given problem and enhance further the accuracy and efficiency of the finite element method (FEM) analysis [1]–[3].

The choice of element-type for geometrical modeling in FEM generally involves a trade-off between the flexibility of the element at modeling different geometries and its mathematical complexity. Bricks and tetrahedra [4], for instance, are simple to implement and their parameters are fast to compute. On the other hand, geometrical flexibility of these elements is poor and, because of their straight edges and planar sides, modeling of complex curved structures becomes exceedingly cumbersome and requires extremely fine meshes in order to achieve a satisfactory level of geometrical approximation. This inevitably leads to the reduction of element sizes, i.e., to small-domain (subdomain) techniques. The number of unknowns (unknown field-distribution coefficients) needed to obtain results of satisfactory accuracy becomes very large even for the structures of low and moderate complexity. Accordingly, the requirements in computational resources and computational time are enormous.

Our goal in this paper is to investigate the accuracy of the geometrical models obtained with the large Lagrange-type curved parametric hexahedra of higher (theoretically arbitrary) orders and analyze the influence of the geometry of hexahedra on the accuracy of the FEM solutions.

II. Theoretical Background

The Lagrange-type curved parametric hexahedron is analytically described as

$$\mathbf{r}(u,v,w)=\sum_{i=1}^{M}\mathbf{r}_i\hat{L}_i^{K_{uvw}}(u,v,w) \qquad -1\le u,v,w\le 1, \quad (1)$$

where $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_M$ are the position vectors of the interpolation nodes, and $\hat{L}_i^{K_{uvw}}(u,v,w)$ are the polynomials defined as

$$\hat{L}_i^{K}(u,v,w)=L_m^{K_u}(u)L_n^{K_v}(v)L_l^{K_w}(w),$$
$$i=1+m+n(K_u+1)+l(K_u+1)(K_v+1)^2,$$
$$0\le m\le K_u,\ 0\le n\le K_v,\ 0\le l\le K_w,$$
$$1\le i\le M=(K_u+1)(K_v+1)(K_w+1), \quad (2)$$

with K_u, K_v, and K_w being the adopted geometrical orders of the element along different parametric coordinates. Functions L_m^K are Lagrange interpolating polynomials given by

$$L_m^K(u)=\prod_{\substack{j=0\\ j\ne m}}^{K}\frac{u-u_j}{u_m-u_j}, \quad (3)$$

where u_j are the uniformly spaced interpolating nodes defined on an interval $-1\le u\le 1$, and similarly for

Anđelija Ž. Ilić is with the Vinča Institute of Nuclear Sciences, Laboratory of Physics 010, P.O. Box 522, 11001 Belgrade, Serbia (phone: 381-11-245-4965, 381-11-244-7700; fax: 381-11-444-7963; e-mail: iandja@vin.bg.ac.yu).

Milan M. Ilić is with the School of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11120 Belgrade, Serbia; (e-mail: milanilic@etf.bg.ac.yu).

Branislav M. Notaroš is with the ECE Department, Colorado State University, 1373 Campus Delivery, Fort Collins, CO 80523-1373, USA; (e-mail: notaros@colostate.edu).

$L_n^K(v)$ and $L_l^K(w)$. Equations (1)-(3) define a mapping from the cubical parent domain to the generalized hexahedron, as illustrated in Fig. 1.

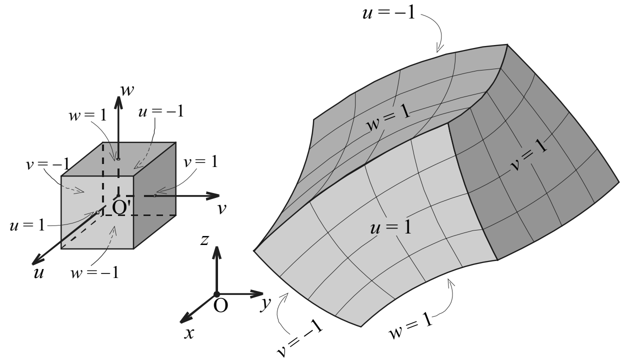

Fig.1. Cube to hexahedron mapping defined by Eqs. (1)-(3).

The electric fields inside the hexahedra are represented by the following expansion:

$$\mathbf{E} = \sum_{i=0}^{N_u-1} \sum_{j=0}^{N_v} \sum_{k=0}^{N_w} \alpha_{uijk}\, \mathbf{f}_{uijk}(u,v,w) + \sum_{i=0}^{N_u} \sum_{j=0}^{N_v-1} \sum_{k=0}^{N_w} \alpha_{vijk}\, \mathbf{f}_{vijk}(u,v,w) + \sum_{i=0}^{N_u} \sum_{j=0}^{N_v} \sum_{k=0}^{N_w-1} \alpha_{wijk}\, \mathbf{f}_{wijk}(u,v,w) \quad (4)$$

$$\mathbf{f}_{uijk}(u,v,w) = f_{uijk}(u,v,w)\, \mathbf{a}_u^r(u,v,w)$$
$$\mathbf{f}_{vijk}(u,v,w) = f_{vijk}(u,v,w)\, \mathbf{a}_v^r(u,v,w)$$
$$\mathbf{f}_{wijk}(u,v,w) = f_{wijk}(u,v,w)\mathbf{a}_w^r(u,v,w) \quad (5)$$

$$\mathbf{a}_u'' = \mathbf{a}_v \times \mathbf{a}_w,\quad \mathbf{a}_v'' = \mathbf{a}_w \times \mathbf{a}_u,\quad \mathbf{a}_w'' = \mathbf{a}_u \times \mathbf{a}_v,$$
$$\mathbf{a}_u^r = \mathbf{a}_u''/J,\quad \mathbf{a}_v^r = \mathbf{a}_v''/J,\quad \mathbf{a}_w^r = \mathbf{a}_w''/J,$$
$$J = (\mathbf{a}_u \times \mathbf{a}_v)\cdot \mathbf{a}_w, \quad (6)$$

$$\mathbf{a}_u = \frac{\partial \mathbf{r}}{\partial u},\qquad \mathbf{a}_v = \frac{\partial \mathbf{r}}{\partial v},\qquad \mathbf{a}_w = \frac{\partial \mathbf{r}}{\partial w}, \quad (7)$$

where f are curl-conforming hierarchical polynomial basis functions of coordinates u, v, and w, N_u, N_v, and N_w are the adopted field approximation orders, which are entirely independent from the element geometrical orders, α_{uijk}, α_{vijk}, and α_{wijk} are unknown field-distribution coefficients, and $\mathbf{r}$ is given in Eq.(1).

To solve for the coefficients $\{\alpha\}$, the expansion in Eq.(4) is substituted in the curl-curl electric-field vector wave equation

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} = 0, \quad (8)$$

where ε_r and μ_r are complex relative permittivity and permeability of the inhomogeneous (possibly lossy) medium, respectively, $k_0 = \omega\sqrt{\varepsilon_0 \mu_0}$ is the free-space wave number, and ω is the angular frequency of the implied time-harmonic variation. A standard Galerkin-type weak form discretization of Eq.(8) yields

$$\int_V \mu_r^{-1}(\nabla \times \mathbf{f}_{\hat{i}\hat{j}\hat{k}})\cdot(\nabla\times\mathbf{E})\,\mathrm{d}V - k_0^2 \int_V \varepsilon_r \mathbf{f}_{\hat{i}\hat{j}\hat{k}} \cdot \mathbf{E}\,\mathrm{d}V = -\oint_S \mu_r^{-1} \mathbf{f}_{\hat{i}\hat{j}\hat{k}} \cdot \mathbf{n} \times (\nabla \times \mathbf{E})\,\mathrm{d}S \quad (9)$$

where V is the volume of a generalized hexahedron, $\mathbf{f}_{\hat{i}\hat{j}\hat{k}}$ stands for any of the functions $\mathbf{f}_{u\hat{i}\hat{j}\hat{k}}$, $\mathbf{f}_{v\hat{i}\hat{j}\hat{k}}$ or $\mathbf{f}_{w\hat{i}\hat{j}\hat{k}}$, S is the boundary surface of the hexahedron, and $\mathbf{n}$ is the outward unit normal ($\mathrm{d}\mathbf{S} = \mathbf{n}\mathrm{d}S$). Due to the continuity of the tangential component of the magnetic field intensity vector, $\mathbf{n} \times \mathbf{H}$, and hence the vector $\mathbf{n} \times (\nabla \times \mathbf{E})$ in Eq.(9) across the interface between any two finite elements in the FEM model, the right-hand side term in Eq.(9) contains the surface integral over the overall boundary surface of the entire FEM domain, and not over the internal boundary surfaces between the individual hexahedra in the model. The tangential component of **H** over the boundary surface of the FEM domain is determined by appropriate boundary conditions imposed at the surface. In analysis of metallic cavities, for instance, these conditions reduce to the simple requirement that the tangential component of **E** vanish near the cavity walls, which is enforced by *a priori* setting to zero the coefficients $\{\alpha\}$ associated with the tangential **E** on the sides of elements adjacent to cavity walls.

The simplest class of hierarchical higher order basis functions on generalized hexahedra is a set of simple 3-D power functions in the $u-v-w$ coordinate system modified for curl conformity, that is, to automatically satisfy the continuity condition for the tangential component of **E** across the side shared by finite elements [1]. These functions, are given by

$$f_{uijk}(u,v,w) = u^i \begin{Bmatrix} 1-v, & j=0 \\ v+1, & j=1 \\ v^j-1, & j\ge 2,\text{even} \\ v^j-v, & j\ge 3,\text{odd} \end{Bmatrix} \begin{Bmatrix} 1-w, & k=0 \\ w+1, & k=1 \\ w^k-1, & k\ge 2,\text{even} \\ w^k-w, & k\ge 3,\text{odd} \end{Bmatrix} \quad (10)$$

with analogous expressions for f_{vijk} and f_{wijk} in Eq.(5).

III. Evaluation of the Lagrange-type Finite Elements for Geometrical Modeling

A. Computational Time and the Demand for Resources

Shown in Fig. 2 are four different *hp*-refined higher order FEM models (*h*-refinement stands for the utilization of the smaller elements in the geometrical mesh, whereas *p*-refinement denotes the increase in accuracy by increasing the orders of field expansion functions). Each of the four *h*-refined models is subsequently gradually *p*-refined to yield the same accuracy (throughout the considered frequency range of 8 to 15 GHz) as the simple three-element model. All of the models, i.e. the required accuracy, are so chosen as to reproduce the results of the S-parameter calculation given in [5]. The comparison of our solution with the reference results is given in Fig. 3. The lower curve represents $|S_{11}|$ and the upper one $|S_{21}|$.

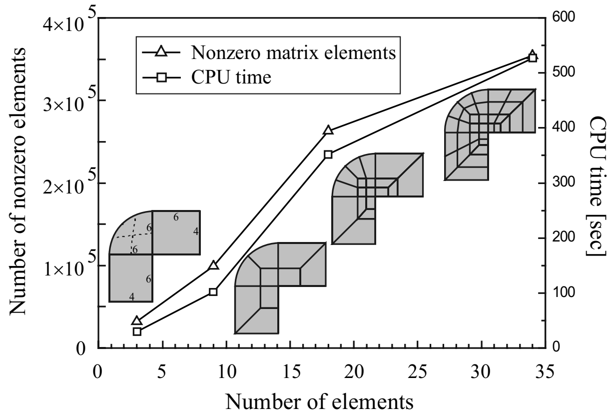

Fig.2. Comparison of the four different *hp*-refined higher order FEM models, with 3, 9, 18, and 34 elements, yielding the same accuracy.

We can observe that the required CPU time and the number of nonzero matrix elements increase with the number of elements comprising the large-domain mesh. It is therefore desirable to use the least number of large elements that yields the required accuracy. It is important to mention at this point that each of the *h*-refined models sets the upper bound on achievable accuracy and although in this particular example three-element model is the optimal one, it is the combined *hp*-refinement that should be used for the best results.

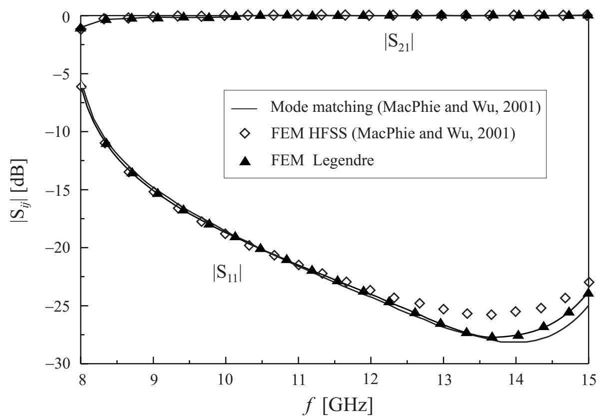

Fig.3. Obtained result shows very good agreement with the ones from reference [5].

B. Accuracy of the curvature modeling

Next, we want to analyze in what amount the inexactness of the geometrical model affects the accuracy of the FEM solution. Fig. 4 shows the relative error in calculation of the effective relative permittivity of an empty circular waveguide. (The exact, analytical, solution has been used to evaluate errors and compare the 2nd and the 4th order model). The 2nd order model ($K_u = 2K_v = 2$) already yields a relative error that is a fraction of percent. With employment of the 4th order model ($K_u = 4K_v = 4$), a significant additional improvement in accuracy can be observed. The *p*-refinement in this model brings the analysis error quickly down below 10^{-4}.

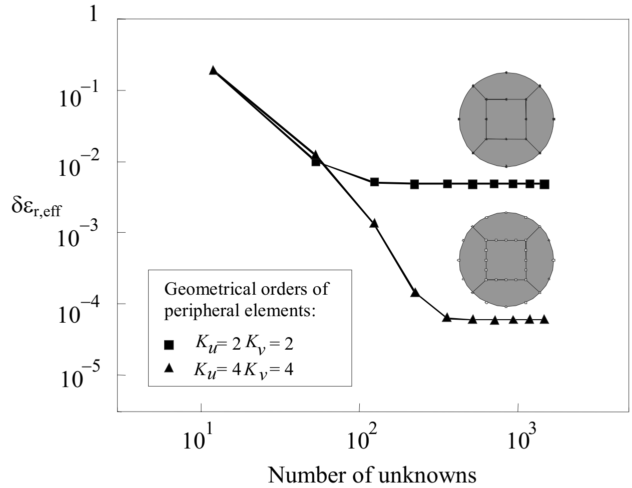

Fig.4. Second and fourth order models as an illustration of the achievable accuracy limits.

We can deduce, however, that it is impossible to *p*-refine any of the models further below the lower bound set by the inherent geometrical errors of the models used to represent the circular cross-section of the waveguide.

IV. Conclusion

The accuracy of the large-domain geometrical models of the curved structures, employing Lagrange-type elements, has been analyzed. We have shown that it is desirable to use the least number of large elements yielding the required accuracy. The error of the geometrical approximation of the circle by large Lagrange-type elements results in the lower bounds set on the calculation error being inherent to the considered circular waveguide models. With the increase in the geometrical orders of elements, as expected, the achievable accuracy also increases. Further mesh refinement can be used as needed, in order to eliminate potentially ill situations, while still keeping the elements as large as possible.

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